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Exact solutions of a three-dimensional nonlinear Schrödinger equation applied to gravity waves

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The three-dimensional evolution of packets of gravity waves is studied using a nonlinear Schrödinger equation (the Davey–Stewartson equation). It is shown that permanent wave groups of the elliptic cn and dn functions and their common limiting solitary sech forms exist and propagate along directions making an angle less than $\psi_c = \tan^{-1}(1/\sqrt{2}) = 35°$ with the underlying wave field, whilst, along directions making an angle greater than ψ_c, there exist permanent wave groups of elliptic sn and negative solitary tanh form. Furthermore, exact general solutions are given showing wave groups travelling along the two characteristic directions at ψ_c or $-\psi_c$. These latter solutions tend to form regions of large wave slope and are used to discuss the waves produced by a ship, in particular the nonlinear evolution of the leading edge of the pattern.

1. Introduction

In recent years, much attention has been given to the problem of the nonlinear evolution and the interaction of progressive waves of slowly varying amplitude and phase moving under gravity in deep water. It was studied by Lighthill (1967) and Hayes (1973) using Whitham's theory, and by Chu & Mei (1970, 1971), Hasimoto & Ono (1972) and Davey & Stewartson (1974) using the techniques of multiple scales. The apparent differences resulting from the two different approaches for two-dimensional waves were resolved by Yuen & Lake (1975) who showed that the nonlinear Schrödinger equation governing the evolution of wave packets as derived using the multiple scales techniques is obtainable by higher-order Whitham theory. They also demonstrated good agreement between theory and experiments.

For three-dimensional packets of surface gravity waves the equation governing the slow evolution of amplitude and phase was first derived by Davey & Stewartson (1974) using the multiple scales technique. It turned out to be a three-dimensional nonlinear Schrödinger equation and was successfully applied by Longuet-Higgins (1976) in the calculation of the nonlinear energy transfer in a narrow gravity wave spectrum of a random sea. He showed that energy from an isolated peak in the spectrum tends to spread outwards along two characteristic lines in wavenumber space, making angles $\psi_c = \tan^{-1}(1/\sqrt{2}) = 35°$ or $-\psi_c$ with the underlying wave field.

In this paper we derive classes of exact solutions to the Davey–Stewartson equation and apply them to study the nonlinear evolution of the wave pattern produced by a

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moving ship. For packets of waves of small surface slope but slowly varying amplitude and phase, we may write the height ζ of the free surface above its undisturbed value in the form

$$\zeta = \operatorname{Re} \epsilon A \exp[i(k_1 x + k_2 y - \bar{\sigma} t)], \tag{1}$$

where t denotes time, and x, y orthogonal horizontal space co-ordinates. The exponential term in (1) represents an underlying carrier wave whose wavenumber (k_1, k_2) and frequency $\bar{\sigma}$ satisfy the linear dispersion relation for irrotational gravity waves in deep water, namely

$$\bar{\sigma}^2 = g(k_1^2 + k_2^2)^{\frac{1}{2}} \tag{2}$$

and A represents a slowly varying amplitude and phase. The (small) parameter characterizes the surface slope, and g is the acceleration due to gravity. For convenience, we choose units of time and length so that

$$g = 1, \quad \bar{\sigma} = 1, \quad (k_1, k_2) = (1, 0). \tag{3}$$

Accordingly, the phase velocity and the group velocity of the carrier waves are 1 and $\frac{1}{2}$, respectively, in the positive x direction. It can be shown that for gravity waves (Davey & Stewartson, 1974) on deep water the evolution of A is, to the first order in ϵ, governed by the partial differential equation†

$$2iA_\tau = \tfrac{1}{4}(A_{\xi\xi} - 2A_{\eta\eta}) + |A|^2 A, \tag{4}$$

where the scaled variables ξ, η and τ are

$$\xi = \epsilon(x - \tfrac{1}{2}t), \quad \eta = \epsilon y, \quad \tau = \epsilon^2 t. \tag{5}$$

The form of the scaling (5) implies an assumption that the length scales for the variation of A in any horizontal direction is of order ϵ^{-1}. From the governing equation it then follows that the time-scale for the evolution of the envelope is of order ϵ^{-2}.

The Davey–Stewartson equation (4) is a form of three-dimensional nonlinear Schrödinger equation. With the substitution

$$A = Re^{i\Theta}, \tag{6}$$

where R and Θ are real functions, (4) is equivalent to a pair of real equations

$$8R_\tau = 2(R_\xi \Theta_\xi - 2R_\eta \Theta_\eta) + R(\Theta_{\xi\xi} - 2\Theta_{\eta\eta}), \tag{7a}$$

and

$$8R\Theta_\tau = R(\Theta_\xi^2 - 2\Theta_\eta^2) - (R_{\xi\xi} - 2R_{\eta\eta}) - 4R^3. \tag{7b}$$

To avoid discontinuity in Θ, R in (6) is defined so that it can change sign. According to (1) and (6), the surface elevation varies within the envelope consisting of $|R|$ and $-|R|$.

As in the two-dimensional case, the Davey–Stewartson equation (4) as derived via the multiple scales technique is also obtainable from higher-order Whitham theory. In fact, the averaged Lagrangian $\mathscr{L}$ in the Whitham variational principle in the three-dimensional case is

$$\mathscr{L} = a^2(\omega^2/\kappa - 1) - \tfrac{1}{2}a^4\kappa^2 + a_t^2\kappa^{-1} + \tfrac{1}{2}(2\bar{k}^2 - \bar{l}^2)\kappa^{-4}a_x^2 + \tfrac{1}{2}(2\bar{l}^2 - \bar{k}^2)\kappa^{-4}a_y^2 + 2\bar{k}\kappa^{-\frac{5}{2}}a_t a_x + 2\bar{l}\kappa^{-\frac{5}{2}}a_t a_y + 3\bar{k}\bar{l}\kappa^{-4}a_x a_y, \tag{8}$$

where $\kappa = (\bar{k}^2 + \bar{l}^2)^{\frac{1}{2}}$ and $a = \epsilon R$,

† Suffixes denote partial differentiation.

a being the amplitude and $\bar{k}$ and $\bar{l}$ the wavenumber components related to the frequency ω through the usual phase function θ,

$$\omega = -\theta_t, \quad \bar{k} = \theta_x \quad \text{and} \quad \bar{l} = \theta_y.$$

In appendix A we show how with the assumption that the frequency and wavenumber depart little from some mean values [this is consistent with the approximations already made in obtaining (8)] the variational principle (8) can be written in the form

$$\delta \iiint [-4R^2\Theta_\tau - R^4 + \tfrac{1}{2}(R_\xi^2 + R^2\Theta_\xi^2) - (R_\eta^2 + R^2\Theta_\eta^2)]\, dx\, dy\, dt = 0,$$

where
$$\theta = x - t + \Theta,$$

this yields precisely equation (7). Thus all solutions of the Davey–Stewartson equation which satisfy the above assumptions will in fact be close approximations to solutions of the averaged variational approach and thus asymptotically valid despite the multiple scales method which was used in deriving the equations. A lower-order Whitham theory would result in the evolution equations (7) without the terms $(R_{\xi\xi} - 2R_{\eta\eta})$.

In § 2, solutions to (7) are obtained for wave groups of permanent form, while in § 3 general solutions are given for waves whose groups travel along the characteristic directions. Application to the nonlinear evolution of the ship wave pattern is discussed in § 4.

2. Permanent wave groups

In this section we look for solutions to the Davey–Stewartson equation (7) for which the frequency and wavenumber are constant. For simplicity in studying the evolution of water waves we may, without loss of generality, choose the phase plane $\theta \equiv x - t + \Theta = \text{const.}$ to be perpendicular to the x axis so that individual waves are propagating in the x direction. Thus we may write

$$\Theta = k\xi - \sigma\tau \tag{9}$$

with k and σ constants, of $O(1)$. Substituting (9) into (7) yields two equations for R:

$$4R_\tau = kR_\xi \tag{10}$$

and
$$R_{\xi\xi} - 2R_{\eta\eta} - (k^2 + 8\sigma)R + 4R^3 = 0. \tag{11}$$

Now from (10),
$$R = R(\xi + \tfrac{1}{4}k\tau, \eta) \tag{12}$$

and we need only consider (11).

We now further restrict ourselves to finding plane progressive solutions of (11) in the form
$$R = R(X), \quad X = \xi\cos\psi + \eta\sin\psi - U\tau, \tag{13}$$

where
$$U = -\tfrac{1}{4}k\cos\psi \tag{14}$$

and the group progresses in a direction making an angle ψ with the carrier waves.

Substituting (13) into (11) with
$$\beta = \tfrac{1}{2}k^2 + 4\sigma, \tag{15}$$

we have
$$\gamma R'' - 2\beta R + 4R^3 = 0, \tag{16}$$

where
$$\gamma = \cos^2\psi - 2\sin^2\psi. \tag{17}$$

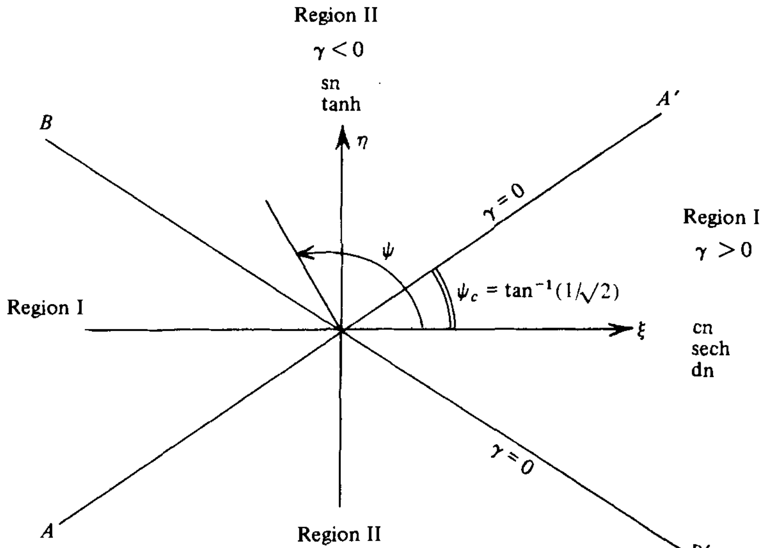

FIGURE 1. Regions for groups of permanent form.

Equation (16) can readily be integrated once, and with the substitution

$$r = R^2 \tag{18}$$

we obtain
$$\gamma(dr/dX)^2 = -8r^3 + 8\beta r^2 + 2Br \equiv K(r), \tag{19}$$

where the constant of integration B is arbitrary.

In order to study the solutions of (19) we divide the $\xi\eta$ plane into four regions by the lines $\gamma = 0$ (see figure 1). Thus, in the two regions I, $\gamma > 0$ whereas, in the two regions II, $\gamma < 0$. We consider first the special case $\gamma = 0$.

2.1. $\tan^2\psi = \frac{1}{2}$

Along the characteristic lines AA' or BB' for which $\gamma = 0$, there are only trivial solutions to (16), namely, $R = 0$ or $R = (2\beta)^{\frac{1}{2}}$ for $\beta > 0$. The latter is simply a constant amplitude plane progressive wave.

2.2. $\tan^2\psi < \frac{1}{2}$

This corresponds to regions I where $\gamma = \cos^2\psi - 2\sin^2\psi > 0$ with the group shape propagating in a direction making an angle less than $\tan^{-1}(1/\sqrt{2})$ with the wave direction. The solution to (19) depends on the nature of the roots of the cubic equation $K(r) = 0$. Out of all possible combinations of β and B, bounded solutions exist only in the following three cases.

(*a*) $B = b^2 > 0$. In this case (19) becomes

$$\tfrac{1}{8}\gamma(dr/dX)^2 = r(r_2 - r)(r - r_1), \tag{20}$$

where
$$r_2 = \tfrac{1}{2}[\beta + (\beta^2 + b^2)^{\frac{1}{2}}] > 0, \quad r_1 = \tfrac{1}{2}[\beta - (\beta^2 + b^2)^{\frac{1}{2}}] < 0. \tag{21}$$

The solution to (20) is

$$r = r_2 \operatorname{cn}^2[(2/\gamma)^{\frac{1}{2}}\, (\beta^2 + b^2)^{\frac{1}{4}} X], \tag{22}$$

where the modulus m of the elliptic cn function is

$$m = (r_2/(r_2 - r_1))^{\frac{1}{2}}. \tag{23}$$

From (22), we obtain the cnoidal envelope solution

$$R = r_2^{\frac{1}{2}} \operatorname{cn}\{(2/\gamma)^{\frac{1}{2}} (\beta^2+b^2)^{\frac{1}{4}} (\xi \cos\psi + \eta \sin\psi - U\tau)\}. \tag{24}$$

(*b*) $\beta = \alpha^2 > 0$ *and* $B = 0$. This is the limiting solution of case (*a*) as $m \to 1$. Thus we have the solitary solution for the group shape

$$R = \beta^{\frac{1}{2}} \operatorname{sech}\{(2\beta/\gamma)^{\frac{1}{2}} (\xi \cos\psi + \eta \sin\psi - U\tau)\}. \tag{25}$$

As seen from (25) along any given direction $\psi < \psi_c = \tan^{-1}(1/\sqrt{2})$, the width of the solitary wave is inversely proportional to its amplitude. On the other hand the width of the solitary wave and the wavelength of the cnoidal wave (24) for the group shape are proportional to $\gamma^{\frac{1}{2}} = (\cos^2\psi - 2\sin^2\psi)^{\frac{1}{2}}$. Hence they vanish when the characteristic directions $\psi = \pm\psi_c$ are reached.

(*c*) $\beta > 0$ *and* $-\beta^2 < B < 0$. In this case (19) becomes

$$\tfrac{1}{8}\gamma(dr/dX)^2 = r(r_2 - r)(r - r_1), \quad r_1 \leqslant r \leqslant r_2, \tag{26}$$

where $$r_2 = \tfrac{1}{2}[\beta + (\beta^2 - b^2)^{\frac{1}{2}}], \quad r_1 = \tfrac{1}{2}[\beta - (\beta^2 - b^2)^{\frac{1}{2}}], \quad b^2 = -B. \tag{27}$$

The solution to (26) is then

$$r = r_1/[1 - m^2 \operatorname{sn}^2\{(2r_2/\gamma)^{\frac{1}{2}} X\}], \tag{28}$$

where the modulus m of the elliptic sn function is

$$m = [(r_2 - r_1)/r_2]^{\frac{1}{2}}. \tag{29}$$

From (28), we obtain

$$R = r_2^{\frac{1}{2}} \operatorname{dn}\left\{\left(\frac{2r_2}{\gamma}\right)^{\frac{1}{2}} (\xi \cos\psi + \eta \sin\psi - U\tau)\right\}. \tag{30}$$

It should be pointed out that, in the limit $B \to 0$, (30) also becomes the solitary solution in case (*b*). Furthermore, for a given direction ψ the wavelength of the group shape as given by (30) is inversely proportional to its amplitude and vanishes when the characteristic directions are reached.

Summing up, since the constant β and B can be chosen arbitrarily, we conclude that if the angle ψ between the group propagation direction and that of the waves satisfies $\cos^2\psi > 2\sin^2\psi$ there always exist solutions for the group envelope of the elliptic cn form (24) and of the elliptic dn form (30). These solutions represent infinite groups (of permanent waves) whose envelope varies periodically in space and time. Their common limit is the solitary form (25). These are sketched in figure 2 over one period.

2.3. $\tan^2\psi > \frac{1}{2}$

This corresponds to regions II in figure 1 where $\gamma = \cos^2\psi - 2\sin^2\psi < 0$. In this case equation (19) has bounded solutions only when

$$\beta > 0 \quad \text{and} \quad -\beta^2 < B \leqslant 0. \tag{31}$$

We may then write (19) as

$$-\tfrac{1}{8}\gamma(dr/dX)^2 = r(r_1 - r)(r_2 - r) \geqslant 0, \tag{32}$$

where $$r_1 = \tfrac{1}{2}[\beta - (\beta^2 - b^2)^{\frac{1}{2}}] > 0, \quad r_2 = \tfrac{1}{2}[\beta + (\beta^2 - b^2)^{\frac{1}{2}}] > 0, \tag{33}$$

$$b^2 = -B, \quad 0 \leqslant r \leqslant r_1$$

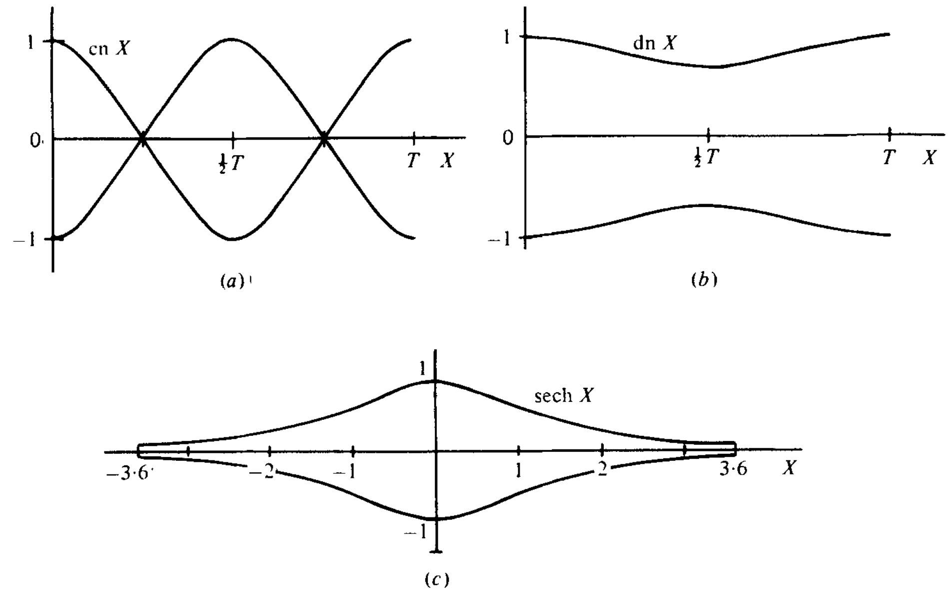

FIGURE 2. Wave group envelopes in region I. (*a*) Infinite periodic cn form, period = T, modulus $m = \frac{1}{2}$. (*b*) Infinite periodic dn form, period = T, $m = \frac{1}{2}$. (*c*) Solitary sech form. The carrier waves in the envelope are progressing in the X direction while the group envelope is progressing in a direction making an angle $\psi < \tan^{-1}(1/\sqrt{2})$ with the plane of the paper.

and the solution is

$$r = r_1\,\mathrm{sn}^2[(-2r_2/\gamma)^{\frac{1}{2}}X], \tag{34}$$

where the elliptic sn function has a modulus

$$m = (r_1/r_2)^{\frac{1}{2}}. \tag{35}$$

The solution for the group amplitude is then

$$R = r_1^{\frac{1}{2}}\,\mathrm{sn}\{(-2r_2/\gamma)^{\frac{1}{2}}\,(\xi\cos\psi + \eta\sin\psi - U\tau)\}, \tag{36}$$

representing an infinite periodic wave group (of permanent waves). An interesting limiting case exists when $b = \beta$, hence $m = 1$ and (36) reduces to

$$R = (\beta/2)^{\frac{1}{2}}\tanh\{(-\beta/\gamma)^{\frac{1}{2}}\,(\xi\cos\psi + \eta\sin\psi - U\tau)\}. \tag{37}$$

The group of (37) has a single depression (figure 3), in direct contrast to the solitary wave (25) in region I which has a single hump. We therefore call (37) a 'negative solitary wave'. It has a sharp trough compared to the round crest in the positive solitary wave case. It is evident that for waves on deep water the negative solitary solution cannot exist for the two-dimensional non-linear Schrödinger equation; it is purely a three-dimensional phenomenon. It can exist however for two-dimensional waves on finite depth if the product of the water depth and the wavenumber of the carrier wave is less than 1·363 (Hasimoto & Ono 1972).

Like the positive solitary wave, the width of the negative solitary wave along any direction $\psi > \psi_c$ is inversely proportional to the depth of its trough. On the other hand the width of the negative solitary wave and the wavelength of the sn wave (36) are proportional to $(2\sin^2\psi - \cos^2\psi)^{\frac{1}{2}}$. Hence, they vanish as the characteristic direction $\psi = \psi_c$ is approached. The solutions (36) and (37) are sketched in figure 3 where, owing to periodicity of (36), only the solution in one period is presented.

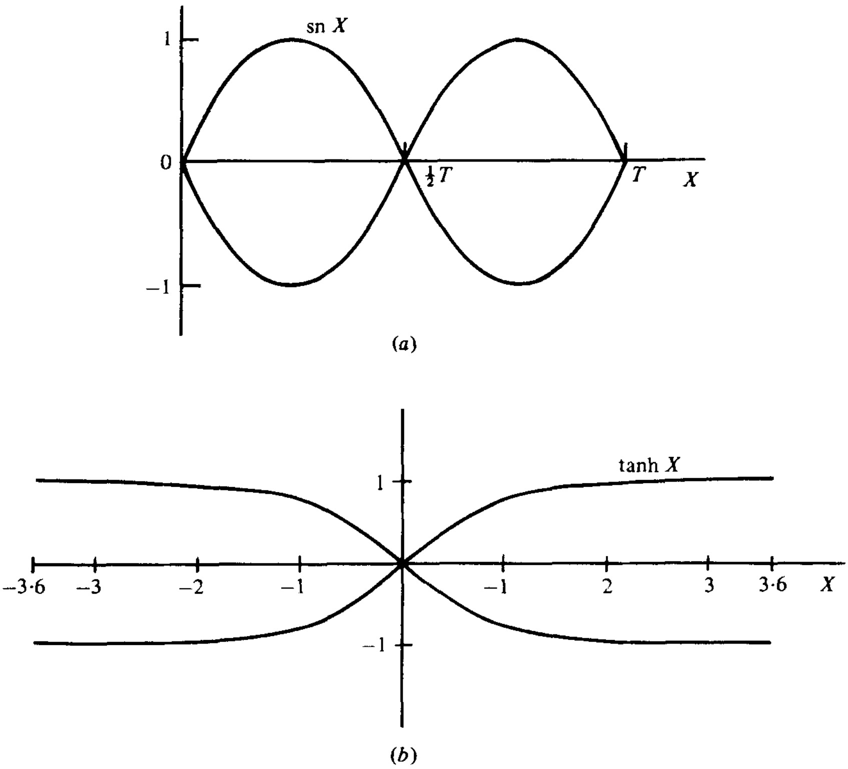

FIGURE 3. Wave group envelopes in region II. (*a*) Infinite periodic sn form, period = T, modulus $m = \frac{1}{2}$. (*b*) Solitary tanh form. The carrier waves in the envelope are progressing in the X direction while the group envelope is progressing in a direction making an angle $\psi > \tan^{-1}(1/\sqrt{2})$ with the plane of the paper.

As an illustration for the negative solitary group of waves, take $\psi = \pi/2$, $\sigma = 0$; then we have a standing group of water waves with a single depression, the surface elevation of which is given by

$$\zeta = a\cos\theta, \quad a = \epsilon\frac{k}{2}\tanh\epsilon\frac{k}{2}y \quad \text{and} \quad \theta = (1+\epsilon k)x - (1+\tfrac{1}{2}\epsilon k)t.$$

2.4. *Discussion*

Firstly, all the solutions obtained in this section have the common property that while the carrier waves propagate in the ξ direction at constant speed, their group shape (the envelope) propagates, also at constant speed, in a different direction making an angle ψ with the ξ axis. This is evidently a three-dimensional property and does not exist in a two-dimensional wave field. As an example, consider (36) and take $\psi = 60°$, $k = \sqrt{2}$, $\sigma = 0$ and $b = \frac{1}{2}\sqrt{3}$, then the surface elevation of this wave group is given by

$$\zeta = \tfrac{1}{2}\epsilon\,\mathrm{sn}\,[\epsilon\sqrt{0{\cdot}3}\,\{x+\sqrt{3}\,y-(\tfrac{1}{2}+\tfrac{1}{8}\sqrt{2}\,\epsilon)\,t\}]\cos[(1+\epsilon\sqrt{2})x-(1+\epsilon/\sqrt{2})t], \qquad (38)$$

where the modulus of the elliptic function is $m = 1/\sqrt{3}$. In (38) the carrier waves are propagating in the x axis while the group envelope is propagating obliquely at an angle 60° to the x axis.

Secondly, we point out that all the solutions obtained above have the corresponding special case, namely the standing group shape waves when $U = 0$. In the two-dimensional case such wave groups were studied by Hasimoto & Ono (1972). (Note 'standing' is relative to the group velocity of the carrier waves.)

Thirdly, we explain the existence of the negative solitary wave in regions II. Equation (16) expresses the balance between the effects of nonlinearity (the term $4R^3$) and dispersion (especially the curvature term $\gamma R''$) as the condition for existence of permanent group shapes. On crossing the characteristic lines γ changes its sign; hence for the curvature to have the same balancing effects the group shape must change from convex to concave, or vice versa. This explains why there is solitary wave on one side of the characteristic line and negative solitary wave on the other side.

In order to get a clearer understanding of the different roles the dispersion plays in the two sides of the characteristic lines, it is instructive to repeat Stokes original construction of wave groups (valid in the small amplitude limit) for general ψ. Thus we start with a small amplitude solution ($\epsilon \to 0$)

$$\zeta = \epsilon R_0 \exp\{i[\bar{k}x - \omega t + d\bar{k}x + d\bar{l}y - (\omega_{\bar{k}}\, d\bar{k} + \omega_{\bar{l}}\, d\bar{l})\, t - (\tfrac{1}{2}\omega_{\bar{k}\bar{k}}\, d\bar{k}^2 + \omega_{\bar{k}\bar{l}}\, d\bar{k}d\bar{l} + \tfrac{1}{2}\omega_{\bar{l}\bar{l}}\, d\bar{l}^2)\, t]\}. \tag{39}$$

Setting $$d\bar{k} = \pm\Delta\cos\psi, \quad d\bar{l} = \pm\Delta\sin\psi, \quad \omega = \bar{k} = 1$$

and adding solutions we obtain

$$\zeta = 2\epsilon R_0 \cos[\Delta(x\cos\psi + y\sin\psi - \tfrac{1}{2}t\cos\psi)]\exp\{i[x - t(1 - \tfrac{1}{4}\Delta^2(\cos^2\psi - 2\sin^2\psi))]\}. \tag{40}$$

Thus if $\cos^2\psi > 2\sin^2\psi$, $|\psi| < |\psi_c|$, the finite group length always decreases the effective frequency (or phase speed), but for $|\psi| > |\psi_c|$ we have the opposite effect.

Finally, we remark on the stability of the wave groups obtained in this section. For some time it has been known that if the effects of finite group length (curvature) are not included, the wave groups as derived from linear theory (Stokes 1847) are unstable when the angle ψ between the propagation velocity of the group and that of the phase is zero (see Benjamin & Feir 1967). This instability may be qualitatively described as the result of higher waves travelling faster and catching up the lower waves, tending to form a discontinuity in the phase (Lighthill 1967). In the three-dimensional case Hayes (1973), using a lower-order Whitham's theory equivalent to (7) without the terms $(R_{\xi\xi} - 2R_{\eta\eta})$, showed that this same instability persists for angles ψ up to the critical angle ψ_c, and for angles greater than ψ_c there is stability. It is, however, not so easy to see what the stability properties would be of the solutions of wave groups of finite group length as given above to the higher-order Whitham's equation (7).

As shown in § 2.1, the only waves of constant frequency/wavenumber type having their group propagating at constant velocity along the characteristic directions $\psi = \pm\psi_c$ are the well-known constant amplitude plane progressive waves. However, other types of waves may still exist whose group propagates along these characteristic directions. To this problem we now turn.

3. Waves having their groups propagating along the characteristic directions

To study waves having their groups propagating along the characteristic directions, it is more convenient to use the characteristic variables μ and ν:

$$\mu = \xi + \eta/\surd 2, \quad \nu = \xi - \eta/\surd 2. \tag{41}$$

The planes μ = const. and ν = const. are perpendicular to the characteristic lines AA' and BB', respectively. Under this transformation, the Davey–Stewartson equations (7) become

$$\left.\begin{aligned} 2R_\tau &= R_\mu \Theta_\nu + R_\nu \Theta_\mu + R\Theta_{\mu\nu} \\ \text{and} \quad 2R\Theta_\tau &= R\Theta_\mu \Theta_\nu - R_{\mu\nu} - R^3. \end{aligned}\right\} \tag{42}$$

3.1. *Derivation of the general solutions*

We now look for classes of solutions to (42) in the form

$$\left.\begin{aligned} R &= F(\mu,\tau) + G(\nu,\tau) \\ \text{and} \quad \Theta &= f(\mu,\tau) + g(\nu,\tau). \end{aligned}\right\} \tag{43}$$

These include as special cases solutions whose group shape propagates in a characteristic direction and in particular any of permanent form.

Substituting (43) into (42) yields

$$\left.\begin{aligned} 2(F_\tau + G_\tau) &= F_\mu g_\nu + G_\nu f_\mu \\ \text{and} \quad 2(f_\tau + g_\tau) &= f_\mu g_\nu - (F+G)^2. \end{aligned}\right\} \tag{44}$$

Differentiation of the first equation of (44) with respect to μ and ν gives

$$F_{\mu\mu} g_{\nu\nu} + f_{\mu\mu} G_{\nu\nu} = 0. \tag{45}$$

From (45) either $F_{\mu\mu} G_{\nu\nu} = 0$ or $F_{\mu\mu} G_{\nu\nu} \neq 0$. In the former case there are only three possibilities, namely (*a*) $G_{\nu\nu} = g_{\nu\nu} = 0$, (*b*) $F_{\mu\mu} = f_{\mu\mu} = 0$, and (*c*) $G_{\mu\mu} = F_{\mu\mu} = 0$. So altogether we need to discuss the following four cases for solutions to (44).

(*a*) $G_{\nu\nu} = g_{\nu\nu} = 0$. In this case

$$G = G_1(\tau)\,\nu + G_0(\tau), \quad g = g_1(\tau)\,\nu + g_0(\tau); \tag{46}$$

substituting (46) into (44) and equating like terms in ν results in

$$G_1 = 0, \quad g_1 = \text{const} \equiv -2U. \tag{47}$$

Hence R is independent of ν and $G_0(\tau)$ can be absorbed into F to render (44) into

$$\left.\begin{aligned} F_\tau + UF_\mu &= 0 \\ \text{and} \quad f_\tau + g_0' &= -Uf_\mu - \tfrac{1}{2}F^2. \end{aligned}\right\} \tag{48}$$

Equations (48) can be solved successively yielding the following solution:

$$\left.\begin{aligned} F &= F(\mu - U\tau) \\ \text{and} \quad f &= -\tfrac{1}{2}\tau F^2(\mu - U\tau) + h(\mu - U\tau) - g_0(\tau). \end{aligned}\right\} \tag{49}$$

We therefore conclude that in this case the general solution to (42) is

$$\left.\begin{aligned} R &= R(\mu - U\tau) \\ \text{and} \quad \Theta &= -\tfrac{1}{2}\tau R^2(\mu - U\tau) - 2U\nu + h(\mu - U\tau). \end{aligned}\right\} \tag{50}$$

This solution contains two arbitrary functions R and h and an arbitrary constant U. The group shape propagates in the μ direction at constant speed ϵU.

(*b*) $F_{\mu\mu} = f_{\mu\mu} = 0$. In this case solutions can be obtained from case (*a*) by interchanging F with G and f with g. Thus we have the following general solutions to (42) with two arbitrary functions R and h,

$$\left.\begin{aligned} R &= R(\nu - U\tau) \\ \text{and} \qquad \Theta &= -\tfrac{1}{2}\tau R^2(\nu - U\tau) - 2U\mu + h(\nu - U\tau). \end{aligned}\right\} \tag{51}$$

The group shape propagates in the ν direction at constant speed ϵU.

(*c*) $F_{\mu\mu} = G_{\nu\nu} = 0$. We may then write

$$F = F_1(\tau)\,\mu + F_0(\tau), \quad G = G_1(\tau)\,\nu + G_0(\tau). \tag{52}$$

Substituting (52) into the first equation of (44) and differentiating the resulting equation with respect to μ and ν respectively, we get

$$\left.\begin{aligned} f &= \frac{F_1'}{G_1}\mu^2 + f_1(\tau)\,\mu + f_0(\tau) \\ \text{and} \qquad g &= \frac{G_1'}{F_1}\nu^2 + g_1(\tau)\,\nu + g_0(\tau) \end{aligned}\right\} \tag{53}$$

with the condition that

$$2(F_0' + G_0') = F_1 g_1 + f_1 G_1.$$

By substituting (52) and (53) into (44) it can easily be shown that in this case (*c*) there are no new solutions other than those found in (*a*) and (*b*).

(*d*) $F_{\mu\mu} G_{\nu\nu} \neq 0$. Then we may write (45) as

$$g_{\nu\nu}/G_{\nu\nu} = -f_{\mu\mu}/F_{\mu\mu} = H_1(\tau)$$

and hence

$$\left.\begin{aligned} f &= -H_1(\tau)\,F + 2H_2(\tau)\,\mu + H_3(\tau) \\ \text{and} \qquad g &= H_1(\tau)\,G + 2H_4(\tau)\,\nu + H_5(\tau), \end{aligned}\right\}$$

where H_i are arbitrary functions of τ. However, further investigations of (44) show that no non-trivial solutions for R and Θ can result in this case.

Summing up, we have shown that for the Davey–Stewartson equation (42) the most general solutions of the form (43) are given by (50) and (51). Their group shape propagates in one of the characteristic directions at a constant speed ϵU relative to the group velocity $\frac{1}{2}$.

3.2. *Discussions*

Firstly, the exact general solutions (50) [or (51)] contain two arbitrary functions R and h and an arbitrary constant U. While ϵU is the speed of propagation of the group shape, $R(\mu)$ and $[h(\mu) - 2U\nu]$ are the initial group envelope and phase, respectively. The fact that R is arbitrary is interesting; it means that wave groups of any envelope form, e.g. a single hump or a single depression, can propagate at constant speed along the characteristic line without changing their shape. On the other hand, this very property that the group envelope shape cannot change during the propagation also rules out the possibility of two or more envelope-solutions passing each other along the characteristic line, in direct contrast to the two-dimensional case for which it is known (see Zakharov & Shabat 1972) that two or more envelope-solitons can pass each other.

Secondly, although the group propagates at constant velocity without changing its shape, the present solution differs from that obtained in § 2 in that the frequency and wavenumber of the underlying waves are no longer constant. In fact the frequency and wavenumber components are given by $-\theta_t$, θ_x and θ_y respectively. Thus, for example,

$$\theta_x = 1+\Theta_x = 1-\epsilon\tau RR'-2\epsilon U+\epsilon h', \quad \text{etc.}, \tag{54}$$

and the x components of the wave slope

$$\partial\zeta/\partial x = \epsilon(\epsilon R'+iR\theta_x)\exp[i(x-t+\Theta)]. \tag{55}$$

It is seen from (54) that the wavenumber and frequency vary with time and space, except in the trivial case $R' = h' = 0$ when we have constant amplitude plane progressive waves. The term $\epsilon\tau RR'$ in (54) also shows that nonlinearities can cumulate to give finite change in wavenumber and frequency. In particular, the wave slope [see (55) and (54)] increases as $-\epsilon^2\tau RR'$. Accordingly, waves tend to become more steepened with time in regions where $R' < 0$ and more flattened where $R' > 0$.

It is also worth noting† that, since U is constant, the term $-2U\nu$ in (50) does not contribute to the time evolution of the underlying carrier waves; it simply changes them by a constant amount, as is seen from (54). Hence to discuss the evolutional aspects of the carrier waves we can put $U = 0$ in (50), which then becomes the solution of equation (4) without the second-order derivative terms $(A_{\xi\xi}-2A_{\eta\eta})$. However, as shown in the special cases in § 2 these second-order derivative terms tend to counterbalance the effects of nonlinearity. But they vanish along the characteristic line, leaving the evolution process controlled entirely by nonlinearity, whence the resulting tendencies of the underlying carrier waves as mentioned in the last paragraph.

Finally, it should be mentioned that wave groups resembling that of (50) or (51) already exist in the Kelvin ship wave pattern in which the direction of propagation of waves and that of their groups make an angle $\tan^{-1}(1/\sqrt{2})$. The next section will be devoted solely to the study of the nonlinear evolution of such a pattern.

4. Applications to the Kelvin ship wave pattern

It is known (e.g. Lamb 1932) that at large distances from a ship moving with uniform velocity **V** the disturbance in the water resolves into: firstly a set of waves transverse to **V** confined between the critical angles $\phi_c = \pm\tan^{-1}(\frac{1}{4}\sqrt{2}) \doteq \pm 19\frac{1}{2}°$ to the ship's course whose amplitude decreases as $t^{-\frac{1}{2}}$, where t measures the age of the wave pattern; secondly, near the critical angles there is a region of waves diverging from **V** of higher amplitude, decreasing according to linear theory only as $t^{-\frac{1}{3}}$ (Hogner 1923; Peters 1949). Ursell (1960) gave a series expansion in terms of the Airy functions for the wave form near these lines (see also Warren 1962 for a simpler first approximation) showing that the crest lengths of these waves increases as $t^{\frac{1}{3}}$.

It is convenient to introduce at this stage the geometric construction due to Lighthill (1957) for determining the wave pattern (figure 4). If OP represents the velocity **V** of the ship, then all waves whose wavenumber components are parallel to OP are the same as a wave of speed $|\mathbf{V}|$ parallel to **V**, have velocities OC say lying on a circle

† We are grateful to the referee for suggesting that a discussion of the general solution (50) along the following lines may be illuminating.

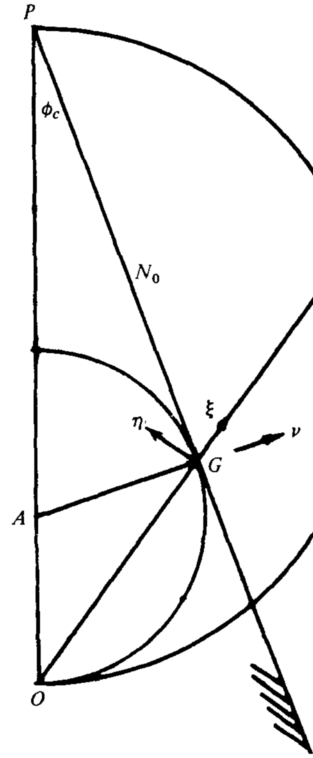

FIGURE 4. Geometry of ship wave pattern.

with OP as diameter. Since the group velocity $OG = \frac{1}{2}OC$ it is simple to construct the critical angle ϕ_c. To make the connexion with the Davey–Stewartson characteristic directions, we point out that the wave near $\phi = \phi_c$ makes an angle equal to AGO with the direction of the 'group' (i.e. the region of amplitude $t^{-\frac{1}{3}}$). This angle is $\tan^{-1}(1/\surd 2)$ and the leading edges of the Kelvin ship wave pattern therefore represent groups travelling in the characteristic directions which may be described by the solution of § 3.

Thus we obtain from Ursell (1960) the following expression for the amplitude of the waves in the neighbourhood of the critical lines

$$\zeta(N,\phi) = \frac{2P}{\rho}\frac{g}{V^4}\frac{p_0(\phi)}{N^{\frac{1}{3}}}\mathrm{Ai}[-N^{\frac{2}{3}}\mu^*(\phi)]\sin[N\nu^*(\phi)] + O(N^{-\frac{2}{3}}). \tag{56}$$

Here P is the total force applied to the water (weight of the vessel), ρ the density of water, g the acceleration due to gravity, $p_0(\phi)$ a function tabulated by Ursell ($p_0(\phi_c) = 3^{\frac{3}{2}}/2^{\frac{3}{2}}$), N the distance from the ship measured in units V^2/g and $\mu^*(\phi)$ and $\nu^*(\phi)$ are given near $\phi = \phi_c$ by

$$\left.\begin{aligned} \mu^*(\phi) &= -3/\surd(2)\,(\phi-\phi_c) + O(\phi-\phi_c)^2 \\ \text{and}\qquad \nu^*(\phi) &= -\surd(3)/2 + \surd(3)/\surd(2)\,(\phi-\phi_c) + O(\phi-\phi_c)^2, \end{aligned}\right\} \tag{57}$$

ϕ being the angle from the ship's path. Now the Davey–Stewartson variables are given by

$$\left.\begin{aligned} \xi &= -\epsilon(N-N_0)\surd(3)/2 + \epsilon N_0(\phi-\phi_c)\surd(3)/\surd(2), \\ \eta &= -\epsilon(N-N_0)\surd(3)/\surd(2) - \epsilon N_0(\phi-\phi_c)\surd(3)/2 \\ \text{and}\qquad \tau &= \epsilon^2 N_0\, 2/\surd(3), \end{aligned}\right\} \tag{58}$$

where the time origin is taken to be when the waves at (N_0, ϕ_c) were generated. Note that

$$N\nu^*(\theta) \doteq -\frac{\sqrt{3}}{2}N_0 - \frac{\sqrt{3}}{2}(N-N_0) + \frac{\sqrt{3}}{\sqrt{2}}N_0(\phi-\phi_c)$$

$$\doteq x - t - \frac{\sqrt{3}}{2}N_0, \tag{59}$$

which is in the direction of increasing ξ, in agreement with the specification of the carrier wave for the Davey–Stewartson equations. Also

$$-N^{\frac{2}{3}}\mu^*(\phi) \doteq N_0^{\frac{2}{3}}\, 3/\sqrt{2}(\phi-\phi_c)$$

$$= \frac{2N_0^{-\frac{1}{3}}}{\epsilon\sqrt{3}}(\xi - \eta/\sqrt{2})$$

$$= \frac{2}{\sqrt{3}}b(\epsilon\tau)\,\nu, \tag{60}$$

here

$$b(\epsilon\tau) = 1/(\epsilon N_0^{\frac{1}{3}}) = (\epsilon\tau\sqrt{(3)}/2)^{-\frac{1}{3}} \tag{61}$$

and ν is the characteristic co-ordinate (41). Now if, near $\tau = \tau^*$ say, we choose $\epsilon = N_0(\tau^*)^{-\frac{1}{3}}$ [N_0 given by (58*c*)] as a measure in agreement with (56) of the wave amplitude, then the variation in the group shape is in agreement with the Davey–Stewartson scaling (5). The Ursell solution (56) may be rewritten as

$$\zeta = \epsilon\zeta_0 p_0\left(\phi_c + \frac{2}{\sqrt{3}}\epsilon^2 b^3(\epsilon\tau)\right) b(\epsilon\tau)\,\mathrm{Ai}\left(\frac{2}{\sqrt{3}}b(\epsilon\tau)\,\nu\right)\sin\left(x - t - \frac{\sqrt{3}}{2}N_0\right), \tag{62}$$

where ζ_0 is a constant.

Thus if $\zeta_0 = O(1)$ or less and for times $\tau = O(\epsilon^{-1})$ or equivalently for points distanced $O(\epsilon^{-3})$ from the ship, the wave pattern in the neighbourhood of the critical lines can be written as the linear form of solution (51) of the Davey–Stewartson equation. Hence setting $b(\epsilon\tau) = 1$ in consistency with the derivation of their equation we may identify

$$R(\nu) = \frac{3^{\frac{3}{2}}Pg}{2^{\frac{1}{2}}\rho V^4}\mathrm{Ai}\left(\frac{2}{\sqrt{3}}\nu\right) \tag{63}$$

and therefore the modifications due to nonlinearity apply. In particular, using the well-known properties (Abramowitz & Stegun 1964) of the Airy function $\mathrm{Ai}(2\nu/\sqrt{3})$ for $\nu < 0$ and the remarks made in § 3.2, it can easily be shown that the slope of the carrier waves will increase with time at the front of the group (on the critical line) and will decrease at the rear. These solutions therefore suggest that wave breaking may occur at the leading edge even for very large time lags.

We note also that although (62) may be identified with the linearized form of (51) whenever $b(\epsilon\tau)$ is a slowly varying function of τ, that is when $\tau = O(1/\epsilon)$, the inclusion of nonlinear terms will cause significant modification to the solution also by times $\tau = O(1/\zeta_0^2)$. Therefore if ζ_0 is not small there is no overlap which would enable Ursell's solution to be used to give initial conditions for the $R(\nu)$.

Since these waves in the neighbourhood of the leading edge vary on the time scale of $1/\epsilon^2$, when $t = O(1/\epsilon^2)$ the inability of the Davey–Stewartson equations to model the linear limit is thought to be probably due to the violation of the assumption of only $O(\epsilon)$ changes in the direction of the waves in different parts of the group.

It may be that the behaviour of the solution of § 3 will also apply in the region $N^2 = O(1/\epsilon^2)$, $\tau = O(1)$ when we would expect a significant nonlinear modification of the linear patterns to have occurred. There is however the further possibility that the divergent waves form a group of nearly permanent form at $\tau = O(1)$, having a constant amplitude or rectangular group shape (§ 2.1). This (if it were stable) would persist to times $O(1/\epsilon^3)$, giving a significantly different aspect to the asymptotic behaviour of the Kelvin ship wave pattern.

Previous investigations into the nonlinear evolution of the Kelvin ship wave pattern include those of Howe (1967) using Whitham theory (without the dispersion terms $R_{\xi\xi} - 2R_{\eta\eta}$ in (7)), Newman (1971) and Hogben (1972). Howe demonstrated the tendency for phase jumps to be generated in the wave pattern, while Newman showed the possibility that the waves near the leading edge be unstable to third-order interactions. Hogben, however, calculated the effects of these interactions using decomposition of the wave pattern into a finite number of modes, and therefore including dispersion. The major effect observed was merely a phase shift in the location of the pattern. Because of the complication of Hogben's analysis and the limited accuracy (due to the finite number of modes employed) it appears that neither instability nor stability have been demonstrated decisively. However, it is clear that nonlinear effects will be important, probably with rapid cumulative effect, and may lead to either of the possibilities suggested above, namely an Airy-function-shaped group at times $O(1/\epsilon^3)$ of nearly constant form with persistent wave breaking at the leading edge, or an approximately rectangular constant width group, established at times $t = O(1/\epsilon^2)$.

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Appendix A. Three-dimensional Whitham equations

It is known that a wave group of arbitrarily small wavenumber can be generated in linear theory merely by the superposition of two (linear) waves of slightly differing frequency/wavenumber (Stokes 1847). Thus it is not surprising that the first-order correction terms for finite group length (a, k, ω not too slowly varying) can be obtained from the linear dispersion relation

$$g(\omega, \bar{k}, \bar{l}) = \omega^2/(\bar{k}^2 + \bar{l}^2)^{\frac{1}{2}} - 1 = 0 \qquad \text{(A 1)}$$

(say) where we are using units such that the acceleration due to gravity is unity. The extended variational principle takes the form (Whitham 1974, p. 526)

$$\delta \iiint \{a^2 g(\omega, \bar{k}, \bar{l}) - \tfrac{1}{2}a^4(\bar{k}^2 + \bar{l}^2) + \tfrac{1}{2}a_t^2\, g_{\omega\omega} + \tfrac{1}{2}a_x^2\, g_{\bar{k}\bar{k}} + \tfrac{1}{2}a_y^2\, g_{\bar{l}\bar{l}} - a_t a_x g_{\omega\bar{k}} - a_t a_y g_{\omega\bar{l}} + a_x a_y g_{\bar{k}\bar{l}}\}\, dx\, dy\, dt = 0. \qquad \text{(A 2)}$$

Yuen & Lake (1975) have shown that (A 2) is strictly correct only with the additional assumption that ω, k, l depart little from their mean values ω_0, k_0, l_0 (say), although the amplitude a may have large cumulative changes. To be specific we have ignored terms such as $a^2k_x^2$ compared with $a_x^2 k^2$. With these assumptions it is consistent to replace ω by $\kappa^{\frac{1}{2}}$ in the higher-order terms of (A 2) $[\kappa = (\bar{k}^2+\bar{l}^2)^{\frac{1}{2}}]$. Thus we have

$$\delta\iiint\{a^2(\omega^2/\kappa-1)-\tfrac{1}{2}a^4\kappa^2+a_t^2\kappa^{-1}+\tfrac{1}{2}(2\bar{k}^2-\bar{l}^2)\kappa^{-4}a_x^2+\tfrac{1}{2}(2\bar{l}^2-\bar{k}^2)\kappa^{-4}a_y^2 + 2\bar{k}\kappa^{-\frac{5}{2}}a_t a_x+2\bar{l}\kappa^{-\frac{5}{2}}a_t a_y+3\bar{k}\bar{l}\kappa^{-4}a_x a_y\}\,dx\,dy\,dt = 0. \quad \text{(A 3)}$$

Reduction to the Davey–Stewartson equation. At this stage the variational principle attaches no special importance to the x or y directions. To obtain an equation of Davey–Stewartson form referred to a carrier wave proceeding in the x direction, we set

$$\omega = 1-\Theta_t, \quad \bar{k} = 1+\Theta_x \quad \text{and} \quad \bar{l} = \Theta_y \quad \text{(A 4)}$$

in (A 3) and neglect higher-order terms to obtain

$$\delta\iiint\{a^2(-2\Theta_t-\Theta_x)-\tfrac{1}{2}a^4+a_t^2+\Theta_t^2a^2+a_x^2+\Theta_x^2a^2-\tfrac{1}{2}a_y^2-\tfrac{1}{2}\Theta_y^2a^2 + 2a_ta_x+2a^2\Theta_t\Theta_x\}\,dx\,dy\,dt = 0. \quad \text{(A 5)}$$

Since the first-order balance is of the form

$$2\,\partial/\partial t+\partial/\partial x = 0$$

we may rewrite (A 4) as

$$\delta\iiint\{-2a^2\Theta_\tau-\tfrac{1}{2}a^4+\tfrac{1}{4}(a_\xi^2+a^2\Theta_\xi^2)-\tfrac{1}{2}(a_y^2+a^2\Theta_y^2)\}\,dy\,d\xi\,dt = 0, \quad \text{(A 6)}$$

where we define $\xi = x-\frac{1}{2}t$. The equations given by variations δa and $\delta\Theta$ in (A 5) may be combined using

$$A = a\,e^{i\Theta} \quad \text{(A 7)}$$

to obtain precisely the Davey–Stewartson equation (4) in § 1. This result that the Davey–Stewartson (three-dimensional nonlinear Schrödinger equation) can be obtained from a variational principle would appear to be new.

Constant form solutions. In looking for constant form solutions it would be possible to proceed directly from (A 6). However the ambiguity in the wave direction (to order ϵ) introduced by (A 4) would remain to hinder the interpretation of the solution. We therefore consider it worth while to derive the group shapes of constant form directly from (A 3); thus we consider solutions of a, ω, $\bar{k}$, $\bar{l}$ as functions of one variable,

$$z = x-ct. \quad \text{(A 8)}$$

From the consistency relations between ω, $\bar{k}$, $\bar{l}$ we obtain

$$\bar{k} = \bar{k}(z), \quad \bar{l} = \text{constant} \quad \text{and} \quad \omega = c\bar{k}+b, \quad \text{(A 9)}$$

where b is an arbitrary constant. With these assumptions (A 3) takes the form

$$\delta\int\left\{a^2((c\bar{k}+b)^2/\kappa-1)-\tfrac{1}{2}a^4\kappa^2+a'^2\left[\frac{c^2}{\kappa}-2\bar{k}c\kappa^{-\frac{5}{2}}+\tfrac{1}{2}(2\bar{k}^2-\bar{l}^2)\kappa^{-4}\right]+\lambda(\bar{k}-\theta')\right\}dz = 0, \quad \text{(A 10)}$$

where λ (= constant) is a Lagrange multiplier. Now we note that if

$$a(z),\ \bar{k}(x),\ \bar{l},\ b,\ \lambda,\ c$$

is a solution, so is

$$\frac{1}{\alpha^2}a(\alpha^2z),\ \alpha^2\bar{k}(\alpha^2z),\ \alpha^2\bar{l},\ \alpha b,\ \lambda/\alpha^6,\ c/\alpha, \quad \text{(A 11)}$$

where α is an arbitrary constant. Hence without loss of generality we may take the group velocity c to be $\frac{1}{2}$. As above, we now replace ω, $\bar{k}$ in the higher-order terms by their mean values, satisfying

$$g(\tfrac{1}{2}k_0+b, k_0, l_0) = 0 \tag{A 12}$$

and, applying variations $\delta\bar{k}$, we obtain

$$\tfrac{1}{2}g_\omega + g_{\bar{k}} = 0 \tag{A 13}$$

which are two simultaneous equations giving k_0, l_0 in terms of the b. Note that if $\bar{k}_0$ satisfies the first-order dispersion relation (A 12) then $\lambda\bar{k}$ must be a second-order quantity and therefore a simple constant playing no part in the variational principle. Thus we set λ to zero.

For the mean values we obtain

$$\kappa_0 = \cos^2\psi, \quad \bar{k}_0 = \kappa_0\cos\psi \quad \text{and} \quad \bar{l}_0 = \kappa_0\sin\psi,$$

where

$$b_0 = \cos\psi - \tfrac{1}{2}\cos^3\psi \tag{A 14}$$

and ψ is the angle between the direction of the waves and the group propagation direction. Thus substituting these values in the higher-order terms of (A 10) and writing

$$g(k_0+\Delta k, b_0+\Delta b) = 2\Delta b/\cos\psi + O(\Delta b^2, \Delta k^2),$$

where $\Delta k(\xi)$ and Δb (a constant) are deviations of $k(\xi)$ and b from the values (A 14), we obtain

$$\delta\int \frac{2a^2\Delta b}{\cos\psi} - \tfrac{1}{2}a\cos^4\psi + \frac{a'^2}{2\cos^2\psi}(\tfrac{1}{2}-\tan^2\psi)\,d\xi = 0. \tag{A 15}$$

A somewhat simpler principle may be obtained by taking $\alpha = 1/\cos\psi$ in (A 11). Thus the mean parameters are then given by

$$\kappa_0^* = 1, \quad \omega_0^* = 1, \quad b_0^* = 1-\tfrac{1}{2}\cos^2\psi, \quad C^* = \tfrac{1}{2}\cos\psi \tag{A 16}$$

and the corresponding amplitude function $a^*(\xi)$ is such that $a^*(\cos^2\psi\xi)/\cos^2\psi$ is a solution of (A 15). Thus a^* satisfies the variational principle

$$\delta\int [2a^{*2}\Delta b^* - \tfrac{1}{2}a^{*4} + \tfrac{1}{2}a^{*\prime 2}(\tfrac{1}{2}\cos^2\psi - \sin^2\psi)]\,d\xi = 0 \tag{A 17}$$

with first integral

$$\frac{\partial}{\partial\xi}[2a^{*2}\Delta b^* - \tfrac{1}{2}a^{*4} - \tfrac{1}{2}a^{*\prime 2}(\tfrac{1}{2}\cos^2\psi - \sin^2\psi)] = 0. \tag{A 18}$$

This equation is equivalent to (16) of § 2.

REFERENCES

ABRAMOWITZ, M. & STEGUN, I. A. (ed.) 1964 *Handbook of Mathematical Functions*, p. 446. New York: Dover.

BENJAMIN, T. B. & FEIR, J. E. 1967 *J. Fluid Mech.* **27**, 417.

CHU, V. H. & MEI, C. C. 1970 *J. Fluid Mech.* **41**, 873.

CHU, V. H. & MEI, C. C. 1971 *J. Fluid Mech.* **47**, 337.

DAVEY, A. & STEWARTSON, K. 1974 *Proc. Roy Soc.* A **338**, 101.

HASIMOTO, H. & ONO, H. 1972 *J. Phys. Soc. Japan* **33**, 805.

HAYES, W. D. 1973 *Proc. Roy Soc.* A **332**, 199.

HOGBEN, N. 1972 *J. Fluid Mech.* **55**, 513.

HOGNER, E. 1923 *Ark. Mat. Astr. Fys.* **17**.

HOWE, M. W. 1967 *J. Fluid Mech.* **32**, 779.

LAMB, H. 1932 *Hydrodynamics.* Cambridge University Press.

LIGHTHILL, M. J. 1957 *River Waves.* Naval Hydrodynamics, publication 515, National Academy of Sciences Research Council.

LIGHTHILL, M. J. 1967 *Proc. Roy. Soc.* A **299**, 25.

LONGUET-HIGGINS, M. S. 1976 *Proc. Roy. Soc.* A **347**, 311.

NEWMAN, J. N. 1971 *J. Ship. Res.* **15**, 1.

PETERS, A. S. 1949 *Com. Pure Appl. Math.* **2**, 123.

STOKES, G. G. 1847 *Camb. Trans.* **8**, 441. (Papers 1, 197–229.)

URSELL, F. 1960 *J. Fluid Mech.* **8**, 418.

WARREN, F. W. G. 1962 *J. Fluid Mech.* **10**, 584.

WHITHAM, G. B. 1974 *Linear and Nonlinear Waves.* Wiley.

YUEN, H. C. & LAKE, B. M. 1975 *Phys. Fluids* **18**, 956.

ZAKHAROV, V. E. & SHABAT, A. B. 1972 *Soviet Phys. J. Exp. Theor. Phys.* **34**, 62.